

Regularisation and the Mullineux map

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Abstract

We classify the pairs of conjugate partitions whose regularisations are images of each other under the Mullineux map. This classification proves a conjecture of Lyle, answering a question of Bessenrodt, Olsson and Xu.

1 Introduction

Suppose $n \geq 0$ and \mathbb{F} is a field of characteristic p ; we adopt the convention that the characteristic of a field is the order of its prime subfield. It is well known that the representation theory of the symmetric group \mathfrak{S}_n is closely related to the combinatorics of partitions. In particular, for each partition λ of n , there is an important $\mathbb{F}\mathfrak{S}_n$ -module S^λ called the *Specht module*. If $p = \infty$, then the Specht modules are irreducible and afford all irreducible representations of $\mathbb{F}\mathfrak{S}_n$. If p is a prime, then for each p -regular partition λ the Specht module S^λ has an irreducible cosocle D^λ , and the modules D^λ afford all irreducible representations of $\mathbb{F}\mathfrak{S}_n$ as λ ranges over the set of p -regular partitions of n .

Given this set-up, it is natural to express representation-theoretic statements in terms of the combinatorics of partitions. An example of this which is of central interest in this paper is the *Mullineux map*. Let sgn denote the one-dimensional sign representation of $\mathbb{F}\mathfrak{S}_n$. Then there is an involutory functor $- \otimes \text{sgn}$ from the category of $\mathbb{F}\mathfrak{S}_n$ -modules to itself. This functor sends simple modules to simple modules, and therefore for each p -regular partition λ there is some p -regular partition $M(\lambda)$ such that $D^\lambda \otimes \text{sgn} \cong D^{M(\lambda)}$. The map M thus defined is now called the Mullineux map, since it coincides with a map defined combinatorially by Mullineux [8]; this was proved by Ford and Kleshchev [3], using an alternative combinatorial description of M due to Kleshchev [5].

Another important aspect of the combinatorics of partitions from the point of view of representation theory is *p -regularisation*. This combinatorial procedure was defined by James in order to describe, for each partition λ , a p -regular partition (which is denoted $G\lambda$ in this paper) such that

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the simple module $D^{G\lambda}$ occurs exactly once as a composition factor of S^λ . In this paper we study the relationship between the Mullineux map and regularisation. Our motivation is the observation that if $p = 2$ or p is large relative to the size of λ , then $MG\lambda = GT\lambda$, where $T\lambda$ denotes the conjugate partition to λ . However, this is not true for arbitrary p , and it natural to ask for which pairs (p, λ) we have $MG\lambda = GT\lambda$. The purpose of this paper is to answer this question, which was first posed by Bessenrodt, Olsson and Xu; the answer confirms a conjecture of Lyle.

If we replace the group algebra $\mathbb{F}\mathfrak{S}_n$ with the Iwahori–Hecke algebra of the symmetric group at a primitive e th root of unity in \mathbb{F} (for some $e \geq 2$), then all of the above background holds true, with the prime p replaced by the integer e (and with an appropriate analogue of the sign representation). Therefore, in this paper, we work with an arbitrary integer $e \geq 2$ rather than a prime p .

In the remainder of this section we give all the definitions we shall need concerning partitions, and state our main result. Section 2 is devoted to proving one half of the conjecture, and Section 3 to the other half. While the first half of the proof consists of elementary combinatorics, the latter half of the proof is algebraic, being an easy consequence of two theorems about v -decomposition numbers in the Fock space. We introduce the background material for this as we need it.

1.1 Partitions

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots$ and the sum $|\lambda| = \lambda_1 + \lambda_2 + \dots$ is finite. We say that λ is a partition of $|\lambda|$. When writing partitions, we usually group together equal parts and omit zeroes. We write \emptyset for the unique partition of 0.

λ is often identified with its *Young diagram*, which is the subset

$$[\lambda] = \{(i, j) \mid j \leq \lambda_i\}$$

of \mathbb{N}^2 . We refer to elements of \mathbb{N}^2 as *nodes*, and to elements of $[\lambda]$ as nodes of λ . We draw the Young diagram as an array of boxes using the English convention, so that i increases down the page and j increases from left to right.

If $e \geq 2$ is an integer, we say that λ is *e-regular* if there is no $i \geq 1$ such that $\lambda_i = \lambda_{i+e-1} > 0$, and otherwise we say that λ is *e-singular*. We say that λ is *e-restricted* if $\lambda_i - \lambda_{i+1} < e$ for all $i \geq 1$.

1.2 Operators on partitions

Here we introduce a variety of operators on partitions. These include regularisation and the Mullineux map, as well as other more familiar operators which will be useful.

1.2.1 Conjugation

Suppose λ is a partition. The *conjugate partition* to λ is the partition $T\lambda$ obtained by reflecting the Young diagram along the main diagonal. That is,

$$(T\lambda)_i = \left| \left\{ j \geq 1 \mid \lambda_j \geq i \right\} \right|.$$

We remark that $T\lambda$ is conventionally denoted λ' ; we choose our notation in this paper so that all operators on partitions are denoted with capital letters written on the left. The letter T is taken from [1], and stands for ‘transpose’.

In this paper we write $l(\lambda)$ for $(T\lambda)_1$, i.e. the number of non-zero parts of λ .

1.2.2 Row and column removal

Suppose λ is a partition. Let $R\lambda$ denote the partition obtained by removing the first row of the Young diagram; that is, $(R\lambda)_i = \lambda_{i+1}$ for $i \geq 1$. Similarly, let $C\lambda$ denote the partition obtained by removing the first column from the Young diagram of λ , i.e. $(C\lambda)_i = \max\{\lambda_i - 1, 0\}$ for $i \geq 1$.

In this paper we shall use without comment the obvious relation $TR = CT$.

1.2.3 Regularisation

Now we introduce one of the most important concepts of this paper. Suppose λ is a partition and $e \geq 2$. The e -regularisation of λ is an e -regular partition associated to λ in a natural way. The notion of regularisation was introduced by James [4] in the case where e is a prime, where it plays a rôle in the computation of the e -modular decomposition matrices of the symmetric groups.

For $l \geq 1$, we define the l th ladder in \mathbb{N}^2 to be the set of nodes (i, j) such that $i + (e - 1)(j - 1) = l$. The regularisation of λ is defined by moving all the nodes of λ in each ladder as high as they will go within that ladder. It is a straightforward exercise to show that this procedure gives the Young diagram of a partition, and the e -regularisation of λ is defined to be this partition.

Example. Suppose $e = 3$, and $\lambda = (4, 3^3, 1^5)$. Then the e -regularisation of λ is $(5, 4, 3^2, 2, 1)$, as we can see from the following Young diagrams, in which we label each node with the number of the ladder in which it lies.

1	3	5	7	
2	4	6		
3	5	7		
4	6	8		
5				
6				
7				
8				
9				

1	3	5	7	9
2	4	6	8	
3	5	7		
4	6	8		
5	7			
6				

We write $G\lambda$ for the e -regularisation of λ . Clearly $G\lambda$ is e -regular, and equals λ if λ is e -regular. We record here three results we shall need later; the proofs of the first two are easy exercises.

Lemma 1.1. Suppose λ is a partition. If $(G\lambda)_1 = \lambda_1$, then $RG\lambda = GR\lambda$.

Lemma 1.2. Suppose λ and μ are partitions. If $l(\lambda) = l(\mu)$ and $GC\lambda = C\mu$, then $G\lambda = G\mu$.

Lemma 1.3. Suppose ζ is an e -regular partition, and $x \geq l(\zeta) + e - 1$. Let ξ be the partition obtained by adding a column of length x to ζ , and let η be the partition obtained by adding a column of length $x - e + 1$ to $C\zeta$. Then $G\eta = CG\xi$.

Proof. For any $n \geq 1$ and any partition λ , let $\text{lad}_n(\lambda)$ denote the number of nodes of λ in ladder n . Since $G\eta$ and $CG\xi$ are both e -regular, it suffices to show that $\text{lad}_n(G\eta) = \text{lad}_n(CG\xi)$ for all n .

η is obtained from ζ by adding the nodes $(l(\zeta) + 1, 1), \dots, (x - e + 1, 1)$, so we have

$$\text{lad}_n(G\eta) = \text{lad}_n(\eta) = \begin{cases} \text{lad}_n(\zeta) + 1 & (l(\zeta) < n < x + e) \\ \text{lad}_n(\zeta) & (\text{otherwise}). \end{cases}$$

It is also easy to compute

$$\text{lad}_n(\xi) = \begin{cases} 1 & (1 \leq n < e) \\ \text{lad}_{n-e+1}(\zeta) + 1 & (e \leq n \leq x) \\ \text{lad}_{n-e+1}(\zeta) & (x < n). \end{cases}$$

Claim. $l(G\xi) = l(\zeta) + e - 1$.

Proof. Since ζ is e -regular and $(l(\zeta), 1) \in [\zeta]$, every node of ladder $l(\zeta)$ is a node of ζ . Hence every node of ladder $l(\zeta) + e - 1$ is a node of ξ ; so when ξ is regularised, none of these nodes moves, and we have $(l(\zeta) + e - 1, 1) \in [G\xi]$, i.e. $l(G\xi) \geq l(\zeta) + e - 1$.

On the other hand, the node $(l(\zeta) + 1, 2)$ does not lie in $[\xi]$, so the node $(l(\zeta) + e, 1)$ cannot lie in $[G\xi]$, i.e. $l(G\xi) < l(\zeta) + e$.

From the claim we deduce that

$$\text{lad}_n(CG\xi) = \begin{cases} \text{lad}_{n+e-1}(\xi) - 1 & (n \leq l(\zeta)) \\ \text{lad}_{n+e-1}(\xi) & (n > l(\zeta)), \end{cases}$$

and combining this with the statements above gives the result. \square

1.2.4 The Mullineux map

Now we introduce the Mullineux map, which is the most important concept of this paper. We shall give two different recursive definitions of the Mullineux map: the original definition due to Mullineux [8], and an alternative version due to Xu [9].

Suppose λ is a partition, and define the *rim* of λ to be the subset of $[\lambda]$ consisting of all nodes (i, j) such that $(i + 1, j + 1) \notin \lambda$. Now fix $e \geq 2$, and suppose that λ is e -regular. Define the e -*rim* of λ to be the subset $\{(i_1, j_1), \dots, (i_r, j_r)\}$ of the rim of λ obtained by the following procedure.

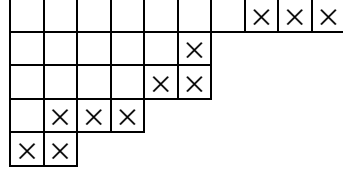
- If $\lambda = \emptyset$, then set $r = 0$, so that the e -rim of λ is empty. Otherwise, let (i_1, j_1) be the top-rightmost node of the rim, i.e. the node $(1, \lambda_1)$.
- For $k > 1$ with $e \nmid k - 1$, let (i_k, j_k) be the next node along the rim from (i_{k-1}, j_{k-1}) , i.e. the node $(i_{k-1} + 1, j_{k-1})$ if $\lambda_{i_{k-1}} = \lambda_{i_{k-1}+1}$, or the node $(i_{k-1}, j_{k-1} - 1)$ otherwise.
- For $k > 1$ with $e \mid k - 1$, define (i_k, j_k) to be the node $(i_{k-1} + 1, \lambda_{i_{k-1}+1})$.
- Continue until a node (i_k, j_k) is reached in the bottom row of $[\lambda]$ (i.e. with $i_k = l(\lambda)$), and either $j_k = 1$ or $e \mid k$. Set $r = k$, and stop.

Less formally, we construct the e -rim of λ by working along the rim from top right to bottom left, and moving down one row every time the number of nodes we've seen is divisible by e .

The integer r defined in this way is called the e -rim length of λ . We define $I\lambda$ to be the partition obtained by removing the e -rim of λ from $[\lambda]$.

Examples.

1. Suppose $e = 3$, and $\lambda = (10, 6^2, 4, 2)$. Then the e -rim of λ consists of the marked nodes in the following diagram, and we see that $r = 11$ and $I\lambda = (7, 5, 4, 1)$.



2. Suppose $e = 2$, and λ is any 2-regular partition. The 2-rim of λ consists of the last two nodes in each row of $[\lambda]$ (or the last node, if there is only one). Hence when $e = 2$ the operator I is the same as C^2 .

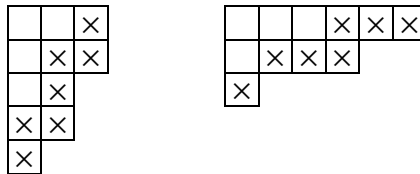
Now we can define the Mullineux map recursively. Suppose λ is an e -regular partition. If $\lambda = \emptyset$, then set $M\lambda = \emptyset$. Otherwise, compute the partition $I\lambda$ as above. Then $|I\lambda| < |\lambda|$, and $I\lambda$ is e -regular, so we may assume that $MI\lambda$ is defined. Let r be the e -rim length of λ , and define

$$m = \begin{cases} r - l(\lambda) & (e \mid r) \\ r - l(\lambda) + 1 & (e \nmid r). \end{cases}$$

It turns out that there is a unique e -regular partition μ which has e -rim length r and $l(\mu) = m$, and which satisfies $I\mu = MI\lambda$. We set $M\lambda = \mu$.

Examples.

1. Suppose $e = 3$, $\lambda = (3^2, 2^2, 1)$ and $\mu = (6, 4, 1)$. Then we have $I\lambda = (2, 1^2)$ and $I\mu = (3, 1)$, as we see from the following diagrams.



Computing e -rim lengths again, we find that $I^2\lambda = I^2\mu = \emptyset$. Now comparing the numbers of non-zero parts of these partitions with their e -rim lengths we find that $MI\lambda = I\mu$, and hence that $M\lambda = \mu$.

2. Suppose $e = 2$, and λ is a 2-regular partition. From above, we see that the 2-rim length of λ is $2l(\lambda)$, if $\lambda_{l(\lambda)} \geq 2$, or $2l(\lambda) - 1$ if $\lambda_{l(\lambda)} = 1$. Either way, we get $m = l(\lambda)$, and this implies inductively that in the case $e = 2$ the Mullineux map is the identity.

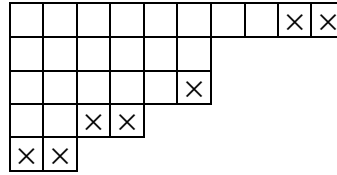
3. Suppose e is large relative to λ ; in particular, suppose e is greater than the number of nodes in the rim of λ . Then the e -rim of λ coincides with the rim, so that the e -rim length is $\lambda_1 + l(\lambda) - 1$. Hence $m = \lambda_1$, and from this it is easy to prove by induction that $M\lambda = T\lambda$.

Now we give Xu's alternative definition of the Mullineux map. Suppose λ is a partition with e -rim length r , and define

$$l' = \begin{cases} l(\lambda) & (e \mid r) \\ l(\lambda) - 1 & (e \nmid r). \end{cases}$$

Define $J\lambda$ to be the partition obtained by removing the e -rim from λ , and then adding a column of length l' . Another way to think of this is to define the *truncated e -rim* of λ to be the set of nodes (i, j) in the e -rim of λ such that $(i, j - 1)$ also lies in the e -rim, together with the node $(l(\lambda), 1)$ if $e \nmid r$, and to define $J\lambda$ to be the partition obtained by removing the truncated e -rim.

Example. Returning to an earlier example, take $e = 3$ and $\lambda = (10, 6^2, 4, 2)$. Then the truncated e -rim of λ consists of the marked nodes in the following diagram, and we see that $l\lambda = (8, 6, 5, 2)$.



If λ is e -regular, then it is a simple exercise to show that $J\lambda$ is e -regular and $|J\lambda| < |\lambda|$. So we assume that $MJ\lambda$ is defined recursively, and we define $M\lambda$ to be the partition obtained by adding a column of length $|\lambda| - |J\lambda|$ to $MJ\lambda$. Xu [9, Theorem 1] shows that this map coincides with Mullineux's map M . In other words, we have the following.

Proposition 1.4. Suppose λ and μ are e -regular partitions, with $|\lambda| = |\mu|$. Then $M\lambda = \mu$ if and only if $MJ\lambda = C\mu$.

1.3 Hooks

Now we set up some basic notation concerning hooks in Young diagrams. Suppose λ is a partition, and (i, j) is a node of λ . The (i, j) -hook of λ is defined to be the set $H_{ij}(\lambda)$ of nodes in $[\lambda]$ directly to the right of or directly below (i, j) , including the node (i, j) itself. The *arm length* $a_{ij}(\lambda)$ is the number of nodes directly to the right of (i, j) , i.e. $\lambda_i - j$, and the *leg length* $l_{ij}(\lambda)$ is the number of nodes directly below (i, j) , i.e. $(T\lambda)_j - i$. The (i, j) -hook length $h_{ij}(\lambda)$ is the total number of nodes in $H_{ij}(\lambda)$, i.e. $a_{ij}(\lambda) + l_{ij}(\lambda) + 1$.

Now fix $e \geq 2$. The e -weight of λ is defined to be the number of nodes (i, j) of λ such that $e \mid h_{ij}(\lambda)$. If $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, we say that $H_{ij}(\lambda)$ is

- *shallow* if $a_{ij}(\lambda) \geq (e - 1)l_{ij}(\lambda)$, or
- *steep* if $l_{ij}(\lambda) \geq (e - 1)a_{ij}(\lambda)$.

Example. Suppose $e = 3$ and $\lambda = (5, 2, 1^4)$. Then we have $(2, 1) \in [\lambda]$, with $a_{2,1}(\lambda) = 1$, $l_{2,1}(\lambda) = 4$, and hence $h_{2,1}(\lambda) = 6$. $H_{2,1}(\lambda)$ is steep if $e = 3$, but not if $e = 6$.

1.4 Lyle's Conjecture

Suppose $e \geq 2$ and λ is an e -regular partition. As noted above, if e is large relative to $|\lambda|$, then $M\lambda = T\lambda$. Of course, there is no hope that this is true in general, since $T\lambda$ will not in general be an e -regular partition. But e -regularisation provides a natural way to obtain an e -regular partition from an arbitrary partition, and it is therefore natural to ask: for which e -regular partitions λ do we have $M\lambda = GT\lambda$? When e is large relative to λ we have $G\lambda = \lambda$ and (from the example above) $M\lambda = T\lambda$, so certainly $M\lambda = GT\lambda$ in this case. We also have $M\lambda = GT\lambda$ for all partitions λ when $e = 2$: we have seen that for $e = 2$ the Mullineux map is the identity, and it is a simple exercise to show that λ and $T\lambda$ have the same 2-regularisation for any λ . But it is not generally true that $M\lambda = GT\lambda$ for an e -regular partition λ . Bessenrodt, Olsson and Xu [1] have given a classification of the partitions for which this does hold, as follows.

Theorem 1.5. [1, Theorem 4.8] *Suppose λ is an e -regular partition. Then $M\lambda = GT\lambda$ if and only if for every $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, the hook $H_{ij}(\lambda)$ is shallow.*

Example. Suppose $e = 4$ and $\lambda = (14, 10, 2^2)$. The Young diagram is as follows; we have marked those nodes (i, j) for which $4 \mid h_{ij}(\lambda)$.

	×		×				×			×				
×		×					×							

We see that all the hooks of length divisible by 4 are shallow, so λ satisfies the second hypothesis of Theorem 1.5. And it may be verified that $GT\lambda = M\lambda = (5^2, 4^2, 3^2, 2^2)$.

Bessenrodt, Olsson and Xu have also posed the following more general question [1, p. 454], which is essentially the same problem without the assumption that λ is e -regular.

For which partitions λ is it true that $M\lambda = GT\lambda$?

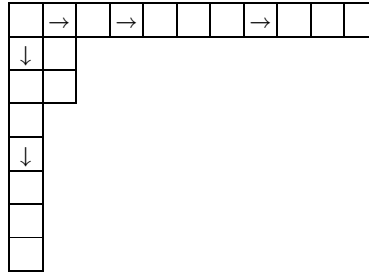
Motivated by the (now solved) problem of the classification of irreducible Specht modules for symmetric groups, Lyle conjectured the following solution in her thesis.

Conjecture 1.6. [7, Conjecture 5.1.18] *Suppose λ is a partition. Then $M\lambda = GT\lambda$ if and only if for every $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$, the hook $H_{ij}(\lambda)$ is either shallow or steep.*

The purpose of this paper is to prove this conjecture. It is a simple exercise to show that a partition possessing a steep hook must be e -singular; so in the case where λ is e -regular, Conjecture 1.6 reduces to Theorem 1.5.

Let us define an L -partition to be a partition satisfying the second condition of Conjecture 1.6, i.e. a partition for which every $H_{ij}(\lambda)$ of length divisible by e is either shallow or steep.

Example. Suppose $e = 4$ and $\lambda = (11, 2^2, 1^5)$. The Young diagram of λ is as follows.



The nodes (i, j) with $4 \mid h_{ij}(\lambda)$ are marked; we see that those marked \rightarrow correspond to shallow hooks, and those marked \downarrow correspond to steep hooks. So λ is an L-partition when $e = 4$. We have $G\lambda = (11, 3, 2^2, 1^2)$, $GT\lambda = (8, 4, 3^2, 2)$, and it can be checked that $MG\lambda = GT\lambda$.

2 The ‘if’ part of Conjecture 1.6

In this section we prove the ‘if’ half of Conjecture 1.6, i.e. that $MG\lambda = GT\lambda$ whenever λ is an L-partition. We begin by noting some properties of L-partitions, and making some more definitions. Note that when $e = 2$, every partition is an L-partition; by the above remarks we have $MG\lambda = GT\lambda$ for every partition when $e = 2$, so Conjecture 1.6 holds when $e = 2$. Therefore, *we assume throughout this section that $e \geq 3$* . The following simple observations will be used without comment.

Lemma 2.1. *Suppose λ is a partition. Then λ is an L-partition if and only if $T\lambda$ is. If λ is an L-partition, then so are $R\lambda$ and $C\lambda$.*

Now we examine the structure of L-partitions in more detail. Suppose λ is an L-partition, and let $s(\lambda)$ be maximal such that $\lambda_{s(\lambda)} - \lambda_{s(\lambda)+1} \geq e$, setting $s(\lambda) = 0$ if λ is e -restricted. Similarly, set $t(\lambda) = 0$ if λ is e -regular, and otherwise let $t(\lambda)$ be maximal such that $(T\lambda)_{t(\lambda)} - (T\lambda)_{t(\lambda)+1} \geq e$. Clearly, we have $s(\lambda) = t(T\lambda)$.

Lemma 2.2. *If λ is an L-partition, then for $1 \leq i \leq s(\lambda)$ we have $\lambda_i - \lambda_{i+1} \geq e - 1$, while for $1 \leq j \leq t(\lambda)$ we have $(T\lambda)_j - (T\lambda)_{j+1} \geq e - 1$.*

Proof. We prove the first statement. Suppose this statement is false, and let $i < s(\lambda)$ be maximal such that $\lambda_i - \lambda_{i+1} < e - 1$. Put $j = \lambda_i - e + 2$. Then we have $(i, j) \in [\lambda]$, with $a_{ij}(\lambda) = e - 2$ and $l_{ij}(\lambda) = 1$, which (given our assumption that $e \geq 3$) contradicts the assumption that λ is an L-partition. \square

Lemma 2.3. *Suppose λ is an L-partition and $(i, j) \in [\lambda]$ with $e \mid h_{ij}(\lambda)$.*

1. *If $i > s(\lambda)$, then $H_{ij}(\lambda)$ is steep.*
2. *If $j > t(\lambda)$, then $H_{ij}(\lambda)$ is shallow.*

Proof. We prove (1). Let $a = a_{ij}(\lambda)$ and $l = l_{ij}(\lambda)$. λ is an L-partition, so if $H_{ij}(\lambda)$ is not steep then it must be shallow, i.e. $a \geq (e-1)l$. In fact, since $e \mid h_{ij}(\lambda) = a + l + 1$, we find that $a \geq (e-1)l + e - 1$. The definition of l implies that $\lambda_{i+l+1} < j = \lambda_i - a$, so

$$\lambda_i - \lambda_{i+l+1} > a \geq (e-1)(l+1),$$

which implies that for some $k \in \{i, \dots, i+l\}$ we have $\lambda_k - \lambda_{k+1} \geq e$. But this contradicts the assumption that $i > s(\lambda)$. \square

Now we define an operator S on L-partitions. Suppose λ is an L-partition, and let $s = s(\lambda)$. Define

$$S\lambda = (\lambda_1 - e + 1, \lambda_2 - e + 1, \dots, \lambda_s - e + 1, \lambda_{s+2}, \lambda_{s+3}, \dots).$$

Note that if λ is an e -restricted L-partition, then $S\lambda = R\lambda$. In general, we need to know that S maps L-partitions to L-partitions, in order to allow an inductive proof of Conjecture 1.6.

Lemma 2.4. *If λ is an L-partition, then so is $S\lambda$.*

Proof. Suppose λ is an L-partition, and that $(i, j) \in [S\lambda]$.

If $i > s(\lambda)$, then $(i+1, j) \in [\lambda]$, and we have

$$a_{ij}(S\lambda) = a_{(i+1)j}(\lambda), \quad l_{ij}(S\lambda) = l_{(i+1)j}(\lambda).$$

So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{(i+1)j}(\lambda)$; so by Lemma 2.3(1) $H_{(i+1)j}(\lambda)$ is steep, and therefore $H_{ij}(S\lambda)$ is steep.

Next suppose $i \leq s(\lambda)$ and $j > \lambda_{s+1}$. Then $(i, j+e-1) \in [\lambda]$ and $a_{ij}(S\lambda) = a_{i(j+e-1)}(\lambda)$, $l_{ij}(S\lambda) = l_{i(j+e-1)}(\lambda)$. So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{i(j+e-1)}(\lambda)$, and so $H_{i(j+e-1)}(\lambda)$ is shallow, and hence $H_{ij}(S\lambda)$ is shallow.

Finally, suppose that $i \leq s(\lambda)$ and $j \leq \lambda_{s+1}$. Then $(i, j) \in [\lambda]$, and we have

$$a_{ij}(S\lambda) = a_{ij}(\lambda) - e + 1, \quad l_{ij}(S\lambda) = l_{ij}(\lambda) - 1.$$

So if $e \mid h_{ij}(S\lambda)$, then $e \mid h_{ij}(\lambda)$, and hence $H_{ij}(\lambda)$ is either shallow or steep. If it is shallow, then we have

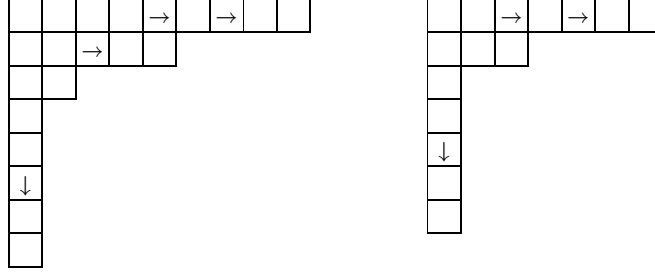
$$a_{ij}(S\lambda) = a_{ij}(\lambda) - e + 1 \geq (e-1)l_{ij}(\lambda) - e + 1 = (e-1)l_{ij}(S\lambda),$$

so that $H_{ij}(S\lambda)$ is shallow. On the other hand, if $H_{ij}(\lambda)$ is steep, then

$$l_{ij}(S\lambda) = l_{ij}(\lambda) - 1 \geq (e-1)a_{ij}(\lambda) - 1 > (e-1)a_{ij}(S\lambda)$$

so $H_{ij}(S\lambda)$ is steep. \square

Example. Suppose $e = 3$, and let $\lambda = (9, 5, 2, 1^5)$. Then we have $s(\lambda) = 2$, so that $S\lambda = (7, 3, 1^5)$. We see that both λ and $S\lambda$ are L-partitions from the following diagrams.



Now we examine the relationship between the operator S and e -regularisation.

Lemma 2.5. *Suppose λ is an L-partition. Then*

$$GTS\lambda = CGT\lambda.$$

Proof. We use induction on $s(\lambda)$. In the case $s(\lambda) = 0$ both λ and $S\lambda = R\lambda$ are e -restricted, i.e. $T\lambda$ and $TS\lambda$ are e -regular, and so $GTS\lambda = TS\lambda = TR\lambda = CT\lambda = CGT\lambda$.

Now suppose $s(\lambda) > 0$. Then $s(R\lambda) = s(\lambda) - 1$, so we may assume that the result holds with λ replaced by $R\lambda$. Put $\zeta = GCT\lambda$; then by the inductive hypothesis $GTSR\lambda = CGTR\lambda = C\zeta$. Let ξ and η be as defined in Lemma 1.3, with $x = \lambda_1$. Note that

$$x = \lambda_1 \geq \lambda_2 + e - 1 = l(CT\lambda) + e - 1 \geq l(GCT\lambda) + e - 1 = l(\zeta) + e - 1,$$

as required by Lemma 1.3.

Claim. $GT\lambda = G\xi$.

Proof. We have $l(T\lambda) = \lambda_1 = l(\xi)$ and $GCT\lambda = \zeta = C\xi$, and Lemma 1.2 gives the result.

Claim. $GTS\lambda = G\eta$.

Proof. Since $s(\lambda) > 0$, $S\lambda$ may be obtained from $SR\lambda$ by adding a row of length $\lambda_1 - e + 1$; hence $TS\lambda$ may be obtained from $TSR\lambda$ by adding a column of length $\lambda_1 - e + 1$. So we have $l(TS\lambda) = \lambda_1 - e + 1 = l(\eta)$, and

$$GCTS\lambda = GTSR\lambda = C\zeta = C\eta,$$

and again we may appeal to Lemma 1.2.

Now Lemma 1.3 combined with these two claims gives the result. □

Next we prove a simple lemma which gives an equivalent statement to the condition $MG\lambda = GT\lambda$ in the presence of a suitable inductive hypothesis.

Lemma 2.6. *Suppose λ is an L-partition, and that $MG\mu = GT\mu$ for all L-partitions μ with $|\mu| < |\lambda|$. Then $MG\lambda = GT\lambda$ if and only if $GS\lambda = JG\lambda$.*

Proof. Since $|G\lambda| = |GT\lambda|$, we have

$$\begin{aligned}
 MG\lambda = GT\lambda &\iff MJG\lambda = CGT\lambda && \text{by Proposition 1.4} \\
 &\iff MJG\lambda = GTS\lambda && \text{by Lemma 2.5} \\
 &\iff MJG\lambda = MGS\lambda && \text{by the inductive hypothesis and Lemma 2.4} \\
 &\iff JG\lambda = GS\lambda. && \square
 \end{aligned}$$

We now require one more lemma concerning the regularisations of L-partitions.

Lemma 2.7. *Suppose λ is an L-partition with $s(\lambda) > 0$ and $\lambda_1 \geq l(\lambda)$. Then:*

1. $(G\lambda)_1 = \lambda_1$;
2. $(G\lambda)_1 - (G\lambda)_2 \geq e - 1$;
3. $(GS\lambda)_1 = (S\lambda)_1$.

Proof.

1. Obviously $(G\lambda)_1 \geq \lambda_1$, so it suffices to show that $[\lambda]$ does not contain a node in ladder $(e-1)\lambda_1 + 1$. If it does, let (i, j) be the rightmost such node. Since $(i, j) \neq (1, \lambda_1 + 1)$, we have $i \geq e$ and we know that the node $(i-e+1, j+1)$ does not lie in λ ; in other words, $(T\lambda)_j - (T\lambda)_{j+1} \geq e$. This means that $j \leq t(\lambda)$, and so by Lemma 2.2 we have $i \leq l(\lambda) - (e-1)(j-1)$, so that

$$l(\lambda) \geq i + (e-1)(j-1) = (e-1)\lambda_1 + 1 > \lambda_1,$$

contrary to hypothesis.

2. By part (1), we must show that $(G\lambda)_2 \leq \lambda_1 - e + 1$, i.e. that $[\lambda]$ does not contain a node in ladder $2 + (e-1)(\lambda_1 - e + 1)$. Supposing otherwise, we let (i, j) be the rightmost such node. Arguing as above, we find that

$$\lambda_1 \geq l(\lambda) \geq i + (e-1)(j-1) = 2 + (e-1)(\lambda_1 - e + 1),$$

and this rearranges to yield $\lambda_1 < e$, which is absurd given that $s(\lambda) > 0$.

3. Obviously $(GS\lambda)_1 \geq (S\lambda)_1 = \lambda_1 - e + 1$, so it suffices to show that $[S\lambda]$ does not contain a node in ladder $1 + (e-1)(\lambda_1 - e + 1)$. Arguing as above, such a node would have to be of the form (i, j) with $j \leq t(S\lambda) \leq t(\lambda)$. But then $(TS\lambda)_j = (T\lambda)_j - 1$, so $[\lambda]$ contains the node $(i+1, j)$, which lies in ladder $2 + (e-1)(\lambda_1 - e + 1)$. But it was shown in (2) that this is not possible. \square

Proof of Conjecture 1.6 ('if' part). We proceed by induction on $|\lambda|$. It is clear that λ is an L-partition if and only if $T\lambda$ is, so Conjecture 1.6 holds for λ if and only if it holds for $T\lambda$. If either λ or $T\lambda$ is e -regular, then the result follows from Theorem 1.5, so we assume that λ is neither e -regular nor e -restricted; in particular, $s(\lambda) > 0$. By replacing λ with $T\lambda$ if necessary, we assume also that $\lambda_1 \geq l(\lambda)$.

Claim. $(JG\lambda)_1 = \lambda_1 - e + 1$, and $RJG\lambda = JGR\lambda$.

Proof. This follows from Lemma 2.7(1–2), given the definition of the operator J .

Claim. $(GS\lambda)_1 = \lambda_1 - e + 1$, and $RGS\lambda = GRS\lambda$.

Proof. We have $(S\lambda)_1 = \lambda_1 - e + 1$ by definition, and $(GS\lambda)_1 = (S\lambda)_1$ by Lemma 2.7(3). The second statement follows from Lemma 1.1.

By induction (replacing λ with $R\lambda$) we have $MGR\lambda = GTR\lambda$, and by Lemma 2.6 (and the inductive hypothesis) this gives $JGR\lambda = GSR\lambda$. Since obviously $GSR\lambda = GRS\lambda$, the two claims yield $JG\lambda = GS\lambda$. Now applying Lemma 2.6 again gives the result. \square

3 The Fock space and v -decomposition numbers

In this section, we complete the proof of Conjecture 1.6 using v -decomposition numbers. We give only a very brief sketch of the background material needed, since this is discussed at length elsewhere; in particular, the article of Lascoux, Leclerc and Thibon [6] is an invaluable source.

Fix $e \geq 2$, let v be an indeterminate over \mathbb{Q} , and let \mathcal{U} be the quantum algebra $U_v(\widehat{\mathfrak{sl}}_e)$ over $\mathbb{Q}(v)$. There is a module \mathcal{F} for this algebra called the *Fock space*, which has a *standard basis* indexed by (and often identified with) the set of all partitions. The submodule generated by the empty partition is isomorphic to the *basic representation* of \mathcal{U} . This submodule has a *canonical* $\mathbb{Q}(v)$ -basis

$$\{G(\mu) \mid \mu \text{ an } e\text{-regular partition}\}.$$

The *v -decomposition numbers* are the coefficients obtained when the elements of the canonical basis are expanded in terms of the standard basis, i.e. the coefficients $d_{\lambda\mu}(v)$ in the expression

$$G(\mu) = \sum_{\lambda} d_{\lambda\mu}(v) \lambda.$$

We shall need to quote two results concerning v -decomposition numbers; one concerning the Mullineux map, and the other concerning e -regularisation. The first of these involves the e -weight of a partition, defined in §1.3.

Theorem 3.1. [6, Theorem 7.2] *Suppose λ and μ are partitions with e -weight w , and that μ is e -regular. Then*

$$d_{(T\lambda)(M\mu)}(v) = v^w d_{\lambda\mu}(v^{-1}).$$

The second result we need requires a definition. Given a partition λ , let $z(\lambda)$ be the number of nodes $(i, j) \in [\lambda]$ such that $e \mid h_{ij}(\lambda)$ and $H_{ij}(\lambda)$ is steep. Now we have the following result.

Theorem 3.2. [2, Theorem 2.2] *For any partition λ ,*

$$d_{\lambda(G\lambda)}(v) = v^{z(\lambda)}.$$

Remark. Note that in [2] an alternative convention for the Fock space is used: our $d_{\lambda\mu}(v)$ is written in [2] as $d_{(T\lambda)(T\mu)}(v)$. Accordingly, the statement of [2, Theorem 2.2] involves shallow hooks rather than steep hooks. We hope that no confusion will result.

Now we combine these theorems. First we note the following obvious result about e -weight and the function z .

Lemma 3.3. *Suppose λ is a partition with e -weight w . Then $T\lambda$ also has e -weight w , and $z(T\lambda)$ equals the number of nodes $(i, j) \in [\lambda]$ such that $e \mid h_{ij}(\lambda)$ and $H_{ij}(\lambda)$ is shallow. Hence λ is an L -partition if and only if $w = z(\lambda) + z(T\lambda)$.*

Now we can complete the proof of Conjecture 1.6.

Proof of Conjecture 1.6 ('only if' part). Suppose $MG\lambda = GT\lambda$, and that λ has e -weight w . Then we have

$$\begin{aligned} v^{z(T\lambda)} &= d_{(T\lambda)(GT\lambda)}(v) && \text{by Theorem 3.2} \\ &= d_{(T\lambda)(MG\lambda)}(v) && \text{by hypothesis} \\ &= v^w d_{\lambda(G\lambda)}(v^{-1}) && \text{by Theorem 3.1} \\ &= v^w \cdot v^{-z(\lambda)} && \text{by Theorem 3.2} \end{aligned}$$

so that $w = z(\lambda) + z(T\lambda)$. Now Lemma 3.3 gives the result. \square

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